1 Lever arms assumption

One of the important assumption done in the MPC model involves the lever arms as following :

$$\frac{d}{dt}(I\omega) \approx I\dot{\omega} = \sum_{i=1}^{n}(r_i - p) \times f_i \approx \sum_{i=1}^{n}(r_i - p^*) \times f_i \tag{1}$$

ω is the angular velocity such as : $\omega = \begin{bmatrix} \dot{\theta} \\ \dot{\phi} \\ \dot{\psi} \end{bmatrix}$. The angular position vector is $\Theta = \begin{bmatrix} \theta \\ \phi \\ \psi \end{bmatrix} = \begin{bmatrix} roll \\ pitch \\ yaw \end{bmatrix}$.

r_i is position of the foot in the optimisation frame. p is the vector position $p = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and p^* is the desired position. This approximation is made to solve the problem more easily. Indeed, by deriving, this term does not appear in the Jacobian. Let's first analyse this approximation in the MPC solver behaviour.

1.1 Derivative of the dynamic

The partial derivative F_x needs to be computed. With the previous assumption, the B matrix depends now on the state vector X_k.

$$X_{k+1} = AX_k + B(X_k)U_k = dynamics(X_k, U_k)$$

where $X_k = \begin{bmatrix} x_k & y_k & z_k & \theta_k & \phi_k & \psi_k & \dot{x}_k & \dot{y}_k & \dot{z}_k & \dot{\theta}_k & \dot{\phi}_k & \dot{\psi}_k \end{bmatrix}^T$ and p is the 3 first parameters that represents the position of the CoM. The command vector is $U^T = \begin{bmatrix} f_{x1} \; f_{y1} \; f_{z1} \dots \; f_{xn} \; f_{yn} \; f_{zn} \end{bmatrix}$ where f_{x1} is the ground reaction force among the x-axis in the local frame of the first foot.

$$F_x = \frac{\partial dynamics}{\partial X_k} = A + \frac{\partial B}{\partial X_k}$$

$$B(X_k) = B(p) = \Delta_t \begin{bmatrix} 0_3 & \dots & 0_3 \\ 0_3 & \dots & 0_3 \\ \frac{1}{m}I_3 & \dots & \frac{1}{m}I_3 \\ I^{-1}[r_1 - p]_\times & \dots & I^{-1}[r_n - p]_\times \end{bmatrix}$$

$[\dots]_\times$ is a 3x3 skew-symmetric matrix that represent the cross products as matrix multiplications.

Let's take the following notation : $b(p) = I^{-1}\sum_{i=1}^{n}(r_i - p) \times f_i$

Then,

$$\frac{\partial B}{\partial X_k} = \Delta_t \begin{bmatrix} & 0_3 & & 0_3 & 0_3 & 0_3 \\ & 0_3 & & 0_3 & 0_3 & 0_3 \\ & 0_3 & & 0_3 & 0_3 & 0_3 \\ \frac{\partial b}{\partial x_k} & \frac{\partial b}{\partial y_k} & \frac{\partial b}{\partial z_k} & 0_3 & 0_3 & 0_3 \end{bmatrix}$$

Now, for example with the x_k variable :

$\frac{\partial b}{\partial x_k} = -I^{-1}\sum_{i=1}^{n} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} f_{xi} \\ f_{yi} \\ f_{zi} \end{bmatrix}$ because $(u \times v)' = u' \times v + u \times v'$ and $\frac{\partial p}{\partial x_k} = \begin{bmatrix} \frac{\partial x_k}{\partial x_k} \\ \frac{\partial y_k}{\partial x_k} \\ \frac{\partial z_k}{\partial x_k} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$